# Density of maximal computability structures
# CCA 2019

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# Outline

1. Computability structures (separable, maximal, dense)
2. Does maximal imply dense?
3. Does dense maximal imply separable?

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# Computability structures

## Definition

Let $(X, d)$ be a metric space and $(x_i)$ a sequence in $X$. We say $(x_i)$ is an *effective sequence* in $(X, d)$ if the function $\mathbb{N}^2 \to \mathbb{R}$

$$(i, j) \mapsto d(x_i, x_j)$$

is recursive.

A finite sequence $x_0, \dots, x_n$ is an *effective finite sequence* if $d(x_i, x_j)$ is a recursive real number for each $i, j \in \{0, \dots, n\}$.

# Computability structures

## Definition

If $(x_i)$ and $(y_j)$ are sequences in $X$, we say $((x_i),(y_j))$ is an *effective pair* in $(X,d)$ and write $(x_i) \diamond (y_j)$ if the function $\mathbb{N}^2 \to \mathbb{R}$,

$$(i,j) \mapsto d(x_i, y_j)$$

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# Computability structures

## Definition

Let $(X, d)$ be a metric space and $(x_i)$ a sequence in $X$. A sequence $(y_i)$ is *computable w.r.t* $(x_i)$ in $(X, d)$ iff there exists a computable $F : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d(y_i, x_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbb{N}$. We write $(y_i) \preceq (x_i)$.

# Computability structures

## Definition

Let $(X, d)$ be a metric space. A set $\mathcal{S} \subseteq X^{\mathbb{N}}$ is a computability structure on $(X, d)$ if the following holds:

1. $(x_i), (y_j) \in \mathcal{S}$, then $(x_i) \diamond (y_j)$
2. if $(x_i) \in \mathcal{S}$ and $(y_j) \preceq (x_i)$, then $(y_j) \in \mathcal{S}$

We say $x$ is a computable point in $\mathcal{S}$ iff $(x, x, \dots) \in \mathcal{S}$.

# Separable computability structures

**Example**

Let $(X, d)$ be a metric space. Let $\alpha : \mathbb{N} \to X$ be an effective sequence which is dense in $X$. We define

$$\mathcal{S}_\alpha = \{(x_i) \mid (x_i) \preceq \alpha\}$$

Then $S_\alpha$ is a computability structure on $(X, d)$.

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## Definition

Let $X \subseteq \mathbb{R}^n$. Let $\mathcal{S}$ be the set of sequences $(x_i)$ in $X$ which are computable in $\mathbb{R}^n$. We say $\mathcal{S}$ is a *canonical computability structure*.

# Separable computability structures

## Definition

A computability structure $\mathcal{S}$ such that there exists a dense sequence $\alpha \in \mathcal{S}$ is called *separable*.

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**Definition**

A computability structure $\mathcal{S}$ such that there exists a dense sequence $\alpha \in \mathcal{S}$ is called *separable*.

**Note**

*Not every computability structure on* $(X, d)$ *is separable!*

# Maximal computability structures

**Definition**

We say $\mathcal{S}$ is a maximal computability structure on $(X, d)$ if there exists no computability structure $\mathcal{T}$ such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$.

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**Note**

*Each computability structure is contained in some maximal computability structure.*

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## Note

*Each computability structure is contained in some maximal computability structure.*

## Note

*Every separable structure is maximal however a maximal structure need not be separable.*

# Dense computability structures

**Definition**

A computability structure $\mathcal{S}$ on $(X, d)$ is *dense* if the set of all computable points in $\mathcal{S}$ is dense in $(X, d)$.

**Theorem**

*Let $X \subseteq \mathbb{R}^n$ be a convex set. Let $\mathcal{S}$ be a maximal computability structure on $X$ with the Euclidean metric. Then $\mathcal{S}$ is dense.*

# Some known results used to prove this

I., Validžić 2016 [4]:

**Proposition**

*Let* $X \subseteq \mathbb{R}^n$ *be such that* $d(x, y)$ *is a recursive number for all* $x, y \in X$*. Then there exists an isometry* $f : \mathbb{R}^n \to \mathbb{R}^n$ *such that each element of* $f(X)$ *is a recursive point.*

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**Theorem**

*Let $X \subseteq \mathbb{R}^n$. Let $\mathcal{M}$ be a maximal computability structure on $(X, d_2)$. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\{f(x_i) \mid (x_i) \in \mathcal{M}\}$ is a canonical computability structure on $f(X)$.*

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**Theorem**

*Let $X \subseteq \mathbb{R}^n$ and let $a_0, \dots, a_k$ be maximal geometrically independent effective sequence in $X$. Then $\mathcal{R}^X_{a_0,\dots,a_k}$ is a unique maximal computability structure on $X$ in which $a_0, \dots, a_k$ are computable points.*

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**Proposition**

*Let $S$ be a circle with a computable radius. Then every maximal computability structure on $S$ is separable.*

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**Proposition**

*Let $X$ be a boundary of a triangle in $\mathbb{R}^2$ with the Euclidean metric. Then there exists a maximal computability structure on $X$ which is not separable.*

**Proposition**

*Let $X$ be a boundary of a triangle in $\mathbb{R}^2$ with the Euclidean metric. Then each dense maximal computability structure on $X$ is separable.*

Moreover, we have the following result:

## Proposition

*Let $X$ be a boundary of a triangle in $(\mathbb{R}^2, d_2)$ with vertices $a$, $b$, $c$. Then the following are equivalent:*

1. *$X$ has a separable computability structure.*
2. *$d_2(a, b)$, $d_2(b, c)$ and $d_2(a, c)$ are computable numbers.*
3. *$X$ has a dense maximal computability structure.*

Even more generally, we have:

## Proposition

*Let $K$ be a nonempty finite simplicial complex of dimension 1 in $\mathbb{R}^n$, i.e. a family of line segments in $\mathbb{R}^n$ and their vertices such that for each two line segments $I, J \in K$ such that $I \neq J$ and $I \cap J \neq \emptyset$ we have $I \cap J = \{v\}$, where $v$ is a vertex of both $I$ and $J$. Suppose that each vertex of $K$ belongs to two distinct line segments of $K$. Let $X = \bigcup_{I \in K} I$. Then each dense maximal computability structure on $X$ is separable.*

## Note

*There exists a maximal computability structure on $X$ which is not dense.*

## Proposition

*Let $X$ be an n-dimensional sphere in $\mathbb{R}^{n+1}$ with radius $r \geq 0$. If $X$ has a dense maximal computability structure, then $X$ has a separable computability structure and $r$ is a computable number.*

**Theorem (B., I. 2014)**

*Let $(X, d, \alpha)$ be a computable metric space which has compact closed balls and the effective covering property. If $S \subseteq X$ is a co-recursively enumerable 1-manifold with finitely many connected components, then $S$ is computable in $(X, d, \alpha)$.*

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**Proposition**

*Let $C \subset \mathbb{R}^2$ be a conic i.e. ellipse, parabola or a hyperbola. Then each dense maximal computability structure on $C$ is separable.*

## Proposition

*There exists $X \subset \mathbb{R}^2$ homeomorphic to a circle and a maximal dense computability structure $\mathcal{M}$ on $X$ which is not separable.*

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# Thank you!