Effective compactness and uniqueness of maximal computability structures
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Outline

1. Computability structures
2. Maximal computability structures
3. Known uniqueness result for subspaces of Euclidean space
4. Uniqueness result for general metric spaces

1 Computability structures

2 Maximal computability structures

3 Known uniqueness result for subspaces of Euclidean space

4 Uniqueness result for general metric spaces

Computability structures

Definition

Let (X,d) be a metric space and (x_i) a sequence in X. We say (x_i) is an *effective sequence* in (X,d) if the function $\mathbb{N}^2 \to \mathbb{R}$

$$(i,j) \mapsto d(x_i, x_j)$$

is recursive.

A finite sequence $x_0, \ldots, x_n$ is an *effective finite sequence* if $d(x_i, x_j)$ is a recursive real number for each $i, j \in \{0, \ldots, n\}$.

Computability structures

Definition

If (x_i) and (y_j) are sequences in X, we say $((x_i), (y_j))$ is an *effective pair* in (X, d) and write $(x_i) \diamond (y_j)$ if the function $\mathbb{N}^2 \to \mathbb{R}$,

$$(i, j) \mapsto d(x_i, y_j)$$

is recursive.

Computability structures

Definition

Let (X, d) be a metric space and (x_i) a sequence in X. A sequence (y_i) is *computable w.r.t* (x_i) in (X, d) iff there exists a computable $F : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d(y_i, x_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbb{N}$. We write $(x_i) \preceq (y_j)$.

Computability structures

Definition

Let (X, d) be a metric space. A set $\mathcal{S} \subseteq X^{\mathbb{N}}$ is a computability structure on (X, d) if the following holds:

1. $(x_i), (y_j) \in \mathcal{S}$, then $(x_i) \diamond (y_j)$
2. if $(x_i) \in \mathcal{S}$ and $(y_j) \preceq (x_i)$, then $(y_j) \in \mathcal{S}$

We say x is a computable point in $\mathcal{S}$ iff $(x, x, \dots) \in \mathcal{S}$.

Computability structures

Example

Let (X, d) be a metric space. Let $\alpha : \mathbb{N} \to X$ be an effective sequence which is dense in X. We define

$$\mathcal{S}_\alpha = \{(x_i) \mid (x_i) \preceq \alpha\}$$

Then S_α is a computability structure on (X, d).

Computability structures

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Definition

A computability structure $\mathcal{S}$ such that there exists a dense sequence $\alpha \in \mathcal{S}$ is called *separable*.

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Definition

A computability structure $\mathcal{S}$ such that there exists a dense sequence $\alpha \in \mathcal{S}$ is called *separable*.

Note

Not every computability structure on (X, d) is separable!

Computability structures

Definition

Let $X \subseteq \mathbb{R}^n$. Let $\mathcal{S}$ be the set of all sequences in X which are recursive in $\mathbb{R}^n$. We call $\mathcal{S}$ the *canonical computability structure* on X.

1 Computability structures

2 Maximal computability structures

3 Known uniqueness result for subspaces of Euclidean space

4 Uniqueness result for general metric spaces

Maximal computability structures

Definition

We say $\mathcal{S}$ is a maximal computability structure on (X, d) if there exists no computability structure $\mathcal{T}$ such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$.

Note

Each computability structure is contained in some maximal computability structure.

Maximal computability structures

Note

If $\mathcal{S}$ is separable then $\mathcal{S}$ is maximal. However, not every maximal structure is separable.

Maximal computability structures

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Example

Let γ be an incomputable real number, $X = \{0, \gamma\}$ and d the Euclidean metric on X. Let $\alpha = (0, 0, 0, \dots)$. Let $\mathcal{T} = \{\alpha\}$. Then $\mathcal{T}$ is maximal, however $\mathcal{T}$ is not separable since α is not dense in X.

Maximal computability structures

Note

If $a_0, \ldots, a_n$ is an effective finite sequence then there exists a maximal computability structure in which $a_0, \ldots, a_n$ are computable points.

Namely,

$$\mathcal{T} = \{(a_0, a_0, \ldots), \ldots (a_n, a_n, \ldots)\}$$

is a computability structure. There is a maximal structure $\mathcal{M}$ such that $\mathcal{T} \subseteq \mathcal{M}$.

Such maximal structure need not be unique!

Maximal computability structures

Question

Let (X, d) *be a metric space. Let* $a_0, \ldots, a_k \in X$. *Let* $\mathcal{M}$ *be a maximal computability structure in which* $a_0, \ldots, a_k$ *are computable. Under which conditions is such* $\mathcal{M}$ *unique?*

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Known uniqueness result for subspaces of Euclidean space

Definition

Let V be a real vector space. Let $a_0, \ldots, a_k$ be vectors in V. We say that $a_0, \ldots, a_k$ are geometrically independent points if $a_1 - a_0, \ldots, a_k - a_0$ are linearly independent vectors.

Known uniqueness result for subspaces of Euclidean space

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Definition

Let V be a real vector space. Let $X \subseteq V$. The largest $k \in \mathbb{N}$ such that that there exist geometrically independent points $a_0, \ldots, a_k \in X$ we call the *affine dimension* of X, and write $\dim X = k$.

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Example

- Let $X = [0, 1]$. Then $\dim X = 1$.
- Let $X = [0, 1] \times [0, 1]$. Then $\dim X = 2$.

Known uniqueness result for subspaces of Euclidean space

The following result about uniqueness of maximal computability structures is known for subspaces of $\mathbb{R}^n$ with the Euclidean metric.

Theorem

Let $X \subseteq \mathbb{R}^n$, $k = \dim X$ and $k \geq 1$. If $a_0, \ldots, a_{k-1}$ is a geometrically independent effective finite sequence on X then there exists an unique maximal computability structure on X in which $a_0, \ldots, a_{k-1}$ are computable points.

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Example

Let (X, d) be such that $X = [0,1] \times [0,1]$ and d the Euclidean metric on X. Then $\dim X = 2$. Let $a_0, a_1 \in X$ be a geometrically independent sequence of points which is effective. Then there exists an unique maximal computability structure in which a_0, a_1 are computable.

Uniqueness result for general metric spaces

Question

What can be said about uniqueness of maximal computability structures for spaces (X, d) *with* $X \subseteq \mathbb{R}^n$ *and* d *which is not the Euclidean metric?*

Uniqueness result for general metric spaces

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What can be said about uniqueness of maximal computability structures for spaces (X, d) with $X \subseteq \mathbb{R}^n$ and d which is not the Euclidean metric?

Note

In the following, we study the metric space (I^2, d_∞) where $I^2 = [0, 1]^2$ and

$$d_\infty((x_1, x_2), (y_1, y_2)) = max(|x_1 - y_1|, |x_2 - y_2|)$$

for each $(x_1, x_2), (y_1, y_2) \in I^2$.

Uniqueness result for general metric spaces

Example

Let $a = (0,0)$, $b = (0,1)$. Does (I^2, d_∞) have a unique maximal computability structure in which points a, b are computable?

Answer: (I^2, d_∞) has at least two such structures. Let S_q be the canonical computability structure on I^2. Then S_q is maximal and a, b are computable in S_q.

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Let e be such that $e = (1, \gamma)$ where γ is an incomputable real $0 < \gamma < 1$. Then a, b, e is an effective finite sequence and there exists a maximal computability structure M such that a, b, e are computable points in M.

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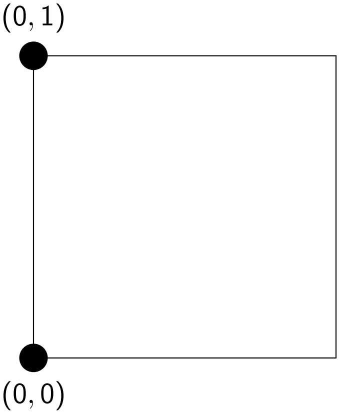
Let e be such that $e = (1, \gamma)$ where γ is an incomputable real $0 < \gamma < 1$. Then a, b, e is an effective finite sequence and there exists a maximal computability structure M such that a, b, e are computable points in M.

However, the point e is not computable in S_q since that would contradict the fact that γ is an incomputable real. This is equivalent to the fact that $(e, e, e, \dots) \notin S_q$. Therefore, $M \neq S_q$.

Uniqueness result for general metric spaces



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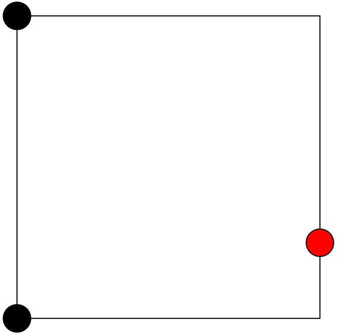


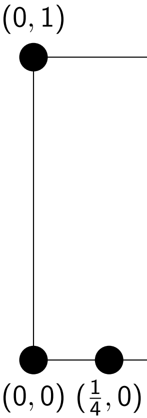
Uniqueness result for general metric spaces

Even choosing three geometrically independent points is not sufficient!

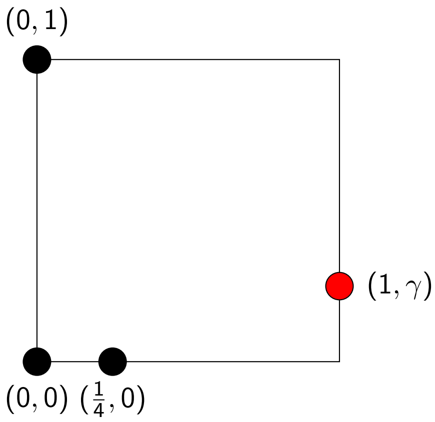
Example

Let $a = (0, 0)$, $b = (0, 1)$ and $c = (\frac{1}{4}, 0)$. Let e be uncomputable like in the previous example. Then a, b, c, e is an effective finite sequence and in the same way we conclude that there are two maximal computability structures in which a, b, c are computable points, namely S_q and $\mathcal{M}$ such that a, b, c, e are computable in $\mathcal{M}$.

Uniqueness result for general metric spaces



Uniqueness result for general metric spaces



Uniqueness result for general metric spaces

Note

Even for the case when geometric independence makes sense, for spaces with non-Euclidean metric the mentioned result about uniqueness of maximal computability structures for subsets of the Euclidean space does not hold.

Uniqueness result for general metric spaces

We wish to introduce for general metric spaces a notion which will be a sort of replacement to the notion of geometric independence.

Definition (Nice sequence)

Suppose (X, d) is a metric space, $n \in \mathbb{N}$ and $a_0, \dots, a_n$ is a finite sequence of points in X such that for all $x, y \in X$ the following implication holds:

$$\text{if } d(a_i, x) = d(a_i, y) \text{ for each } i \in \{0, \dots, n\}, \text{ then } x = y.$$

Then we say that $a_0, \dots, a_n$ is a **nice sequence** in (X, d).

Uniqueness result for general metric spaces

Question

If the finite sequence $a_0, \ldots, a_n$ is nice and effective in (X, d), is then a maximal computability structure $\mathcal{M}$ on (X, d) in which the points $a_0, \ldots, a_n$ are computable, unique?

In general, the answer is negative!

Uniqueness result for general metric spaces

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If the finite sequence $a_0, \ldots, a_n$ is nice and effective in (X, d), is then a maximal computability structure $\mathcal{M}$ on (X, d) in which the points $a_0, \ldots, a_n$ are computable, unique?

In general, the answer is negative!

Example

Let $X = \{a_0, x, y\}$. Let $d(a_0, x) = 1$, $d(a_0, y) = 2$ and $d(x, y) = \gamma$ where $1 < \gamma < 3$ is an incomputable real. Then (X, d) is a metric space and a_0 is nice and effective in (X, d). Let $\mathcal{M}_1$ be a maximal structure such that a_0, x are computable in $\mathcal{M}_1$. Let $\mathcal{M}_2$ be a maximal computability structure in which a_0, y are computable. The point a_0 is computable in both $\mathcal{M}_1$ and $\mathcal{M}_2$, however $\mathcal{M}_1 \neq \mathcal{M}_2$.

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Uniqueness result for general metric spaces

Theorem

Let (X, d) be an effectively compact metric space. Suppose $a_0, \ldots, a_n$ is a nice sequence in (X, d) and suppose that there exists a separable computability structure $\mathcal{S}$ on (X, d) in which $a_0, \ldots, a_n$ are computable points. Then $\mathcal{S}$ is a ***unique*** *maximal computability structure on (X, d) in which $a_0, \ldots, a_n$ are computable points.*

Uniqueness result for general metric spaces

Note

A metric space (X, d) *is said to be effectively compact if there exist an effective separating sequence* α *in* (X, d) *and a computable function* $f : \mathbb{N} \to \mathbb{N}$ *such that*

$$X = B(\alpha_0, 2^{-k}) \cup \cdots \cup B(\alpha_{f(k)}, 2^{-k})$$

for each $k \in \mathbb{N}$*. It is known that if* (X, d) *is effectively compact, then for each effective separating sequence* α *in* (X, d) *there exists such a computable function* f*.*

Uniqueness result for general metric spaces

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A metric space (X, d) is said to be effectively compact if there exist an effective separating sequence α in (X, d) and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

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Proposition

(I^2, d_∞) is effectively compact.

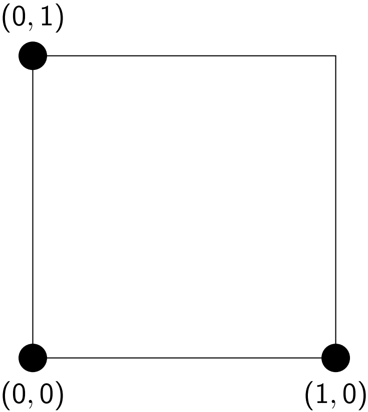
Uniqueness result for general metric spaces

Example

Let S_q be the canonical computability structure on (I^2, d_∞).

Let $a = (0,0)$, $b = (0,1)$ and $c = (1,0)$. Then a, b, c are computable in S_q and a, b, c is a nice sequence in (I^2, d_∞). So, the theorem implies there is a unique maximal computability structure on (I^2, d_∞) such that a, b, c are computable points.

Uniqueness result for general metric spaces



Uniqueness result for general metric spaces

The assumption of nice in the theorem is necessary!

Example

Let $a = (0, 0)$, $b = (0, 1)$, $c = (\frac{1}{4}, 0)$. Then a, b, c are not nice in (I^2, d_∞). We have shown previously that there are at least two maximal computability structures on (I^2, d_∞) in which a, b, c are computable.

Uniqueness result for general metric spaces

Question

Which other sequences a, b, c are nice in (l^2, d_∞)?

Uniqueness result for general metric spaces

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Which other sequences a, b, c *are nice in* (l^2, d_∞)*?*

Proposition

Let $a, b, c \in l^2$ *such that either* $a = (0,0)$*,* $b = (1,1)$ *or* $a = (0,1)$*,* $b = (1,0)$*. Let* $c \notin \overline{ab}$*. Then* a, b, c *is nice in* (l^2, d_∞)*.*

Uniqueness result for general metric spaces

Note

A more general form of the theorem holds: the assumption of effective compactness of the space (X, d) *can be replaced with the assumption that* (X, d) *has compact closed balls and there exists* α *such that* (X, d, α) *has the effective covering property.*

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Thank you!