QR decomposition of split-complex matrices via centrosymmetric representations

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Abstract

The centrosymmetric QR decomposition of a centrosymmetric matrix was introduced previously by the author in [1]. Let A be a square split-complex matrix. We introduce the standard centrosymmetric matrix representation of A denoted by $cs(A)$. We prove that the QR decomposition of A is equivalent to the centrosymmetric QR decomposition of $cs(A)$.

Keywords: QR decomposition, centrosymmetric, split-complex numbers, hyperbolic numbers, representation

1. Introduction

An *algebra* $\mathfrak{A}$ is an ordered pair $(A, \cdot)$ such that A is a vector space over a field K and $\cdot : A \times A \to A$ is a bilinear mapping called *multiplication*.

Let $\mathfrak{D} = (\mathbb{R}^2, \cdot)$, where $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(a, b) \cdot (c, d) = (ac + bd, ad + bc) \tag{1.1}$$

for all $(a, b), (c, d) \in \mathbb{R}^2$. It is straightforward to verify that $\mathfrak{D}$ is an algebra. This is the well known algebra of *split-complex numbers*. The split-complex numbers are also sometimes known as *hyperbolic numbers*. Similarly as for the complex numbers, each real number $x \in \mathbb{R}$ can be identified with the pair $(x, 0)$. With this correspondence, the pair $j \equiv (0, 1)$ has the property

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$j^2 = +1$ and $j \neq \pm 1$. Due to this property, j is called the *hyperbolic unit*. Since $(x, y) = (x, 0) + (0, 1) \cdot (0, y)$, in the following we shall denote a pair (x, y) simply with $x + jy$.

The *conjugate* of $z = a + jb$ is defined as $z^* = a - jb$. For a hyperbolic number $z = a + jb$ we define the *real part* as Re $(z) = a$ and *hyperbolic part* as Hyp $(z) = b$. For the module we set $|z| = zz^* = a^2 - b^2$ and we have $|w \cdot z| = |w| \cdot |z|$ for all $w, z \in \mathfrak{D}$. For an extensive overview of the theory of hyperbolic numbers as well of their usefulness in Physics one can check the literature, for example [2]. For the rest of this paper, we shall refer to these numbers as *split-complex numbers*. By $M_n(\mathfrak{D})$ we denote the set of all $n \times n$ *split-complex matrices* i.e. matrices in which entries are split-complex numbers. Note that $Z \in M_n(\mathfrak{D})$ if and only if there exist $n \times n$ real matrices A, B such that $Z = A + jB$. If A is a matrix then its *transpose* is defined as $a^T_{i,j} = a_{j,i}$ for all i, j and is denoted with A^T. In the following we denote by I_n the $n \times n$ *identity matrix* and by O_n the $n \times n$ *zero matrix*. Let $J \in M_n(\mathbb{R})$ be defined as $J = \text{antidiag}(1, \ldots, 1)$ for each n. Note that $J^2 = I$. A matrix $R \in M_n(\mathbb{R})$ is *upper-triangular* if $r_{i,j} = 0$ for all $i > j$. A real matrix R is *centrosymmetric* if $RJ = JR$. An overview of centrosymmetric matrices can be found in [4]. We denote by $CM_n(\mathbb{R})$ the set of all $n \times n$ centrosymmetric real matrices.

If V is a finite-dimensional vector space then we denote by $\mathcal{L}(V)$ the set of all linear transformations $V \to V$. Recall that if V is an n-dimensional vector space and $\mathcal{B} = (v_1, \ldots, v_n)$ an ordered basis for V, then every linear transformation $T : V \to V$ has an $n \times n$ matrix representation with respect to $\mathcal{B}$ denoted $[T]_{\mathcal{B}}$. Further, for any two linear transformations $S, T : V \to V$ we have $[S \circ T]_{\mathcal{B}} = [S]_{\mathcal{B}}[T]_{\mathcal{B}}$. The standard ordered basis for $\mathbb{R}^n$ i.e. the basis $(e_1, \ldots, e_n)$ is defined as $e_{i,j} = 1$ if $i = j$ and $e_{i,j} = 0$ otherwise.

Let $\mathfrak{A} = (A, \cdot)$ be an algebra. A *representation* of $\mathfrak{A}$ over a vector space V is a map $\phi : A \to \mathcal{L}(V)$ such that $\phi(a_1 \cdot a_2) = \phi(a_1) \circ \phi(a_2)$ for all $a_1, a_2 \in A$. If V is an n-dimensional vector space and $\mathcal{B} = (v_1, \ldots, v_n)$ an ordered basis for V then every linear transformation $T : V \to V$ has a matrix representation $[T]_{\mathcal{B}} \in M_n(\mathbb{R})$. For each $a \in A$ we have $\phi(a) \in \mathcal{L}(V)$. Since V is n-dimensional, we have and ordered basis $\mathcal{B}$ and $[\phi(a)]_{\mathcal{B}} \in M_n(\mathbb{R})$. A *matrix representation of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ is a map $\phi_{\mathcal{B}} : A \to M_n(\mathbb{R})$ such that $\phi_{\mathcal{B}}(a) = [\phi(a)]_{\mathcal{B}}$ for all $a \in A$. Further, we have $\phi_{\mathcal{B}}(a_1 \cdot a_2) = \phi(a_1)_{\mathcal{B}} \cdot \phi(a_2)_{\mathcal{B}}$ for all $a_1, a_2 \in A$. These are well known notions from representation theory, for further information, one can consult one of the standard textbooks, for example see [3].

2. Centrosymmetric representation of split-complex matrices

For the algebra $\mathfrak{D}$ of split-complex numbers the well-known matrix representation $cs : \mathfrak{D} \to M_2(\mathbb{R})$ with respect to (e_1, e_2) is given by

$$cs(z) = \begin{bmatrix} \text{Re } (z) & \text{Hyp } (z) \\ \text{Hyp } (z) & \text{Re } (z) \end{bmatrix}, \qquad \forall z \in \mathfrak{D}. \tag{2.1}$$

It is straightforward to check that for all $w, z \in \mathfrak{D}$ we have $cs(w \cdot z) = cs(w)cs(z)$.

Further, on the vector space $M_n(\mathfrak{D})$ there is a natural multiplication operation $\cdot : M_n(\mathfrak{D}) \times M_n(\mathfrak{D}) \to M_n(\mathfrak{D})$ given by

$$(A + jB) \cdot (C + jD) := (AC + BD) + j(AD + BC) \tag{2.2}$$

for all $A, B, C, D \in M_n(\mathbb{R})$. It is easy to verify that $(M_n(\mathfrak{D}), \cdot)$ is an algebra. In the following we refer to this algebra as the *algebra of split-complex (square) matrices* and denote it with $\mathfrak{M}_n$.

Note that in the following whenever we have two matrices $A, B \in M_n(\mathfrak{D})$, their product shall explicitly be written with a dot '$\cdot$', e.g. $A \cdot B$ to indicate multiplication defined in (2.2). Otherwise, if $A, B \in M_n(\mathbb{R})$ we simply write AB.

To state and prove our main result, we shall need the following well known characterization of centrosymmetric matrices.

Proposition 2.1. *Let $A \in M_{2n}(\mathbb{R})$. Then $A \in CM_{2n}(\mathbb{R})$ if and only if there exist $B, C \in M_n(\mathbb{R})$ such that*

$$A = \begin{bmatrix} B & CJ \\ JC & JBJ \end{bmatrix}. \tag{2.3}$$

Proof. Suppose

$$A = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}.$$

Since A is centrosymmetric, we have $AJ = JA$, or equivalently, in block-form

$$\begin{bmatrix} & J \\ J & \end{bmatrix} \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} \begin{bmatrix} & J \\ J & \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} JW & JZ \\ JX & JY \end{bmatrix} = \begin{bmatrix} YJ & XJ \\ ZJ & WJ \end{bmatrix}$$

We now have $Z = JXJ$ and $Y = JWJ$, so

$$A = \begin{bmatrix} X & JWJ \\ W & JXJ \end{bmatrix}$$

Now, by choosing $C = JW$ and $B = X$ and from the fact $J^2 = I$ it follows that A has the form (2.3).

Conversely, suppose A has the form (2.3). It can easily be shown by block-matrix multiplication that $AJ = JA$, hence A is centrosymmetric. □

Proposition 2.2. *The map* $cs : M_n(\mathfrak{D}) \to CM_{2n}(\mathbb{R})$ *defined as*

$$cs(A + jB) = \begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix} \tag{2.4}$$

is a matrix representation of $\mathfrak{M}_n$*. We call the representation cs the* standard centrosymmetric matrix representation *of* $\mathfrak{M}_n$*.*

Proof. Let $W \in M_n(\mathfrak{D})$ and $Z \in M_n(\mathfrak{D})$ be such that $W = A + jB$ and $Z = C + jD$. We now have

$$\begin{aligned} cs(W)cs(Z) &= \begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix} \begin{bmatrix} C & DJ \\ JD & JCJ \end{bmatrix} \\ &= \begin{bmatrix} AC + BD & (AD + BC)J \\ J(AD + BC) & J(AC + BD)J \end{bmatrix} = cs(W \cdot Z) \end{aligned}$$

which proves the claim. □

Proposition 2.3. *Let* $Q \in M_n(\mathfrak{D})$*. Then* $cs(Q^T) = cs(Q)^T$*.*

Proof. Let $Q = A + jB$. Then

$$cs(Q^T) = cs(A^T + jB^T) = \begin{bmatrix} A^T & B^T J \\ JB^T & JA^T J \end{bmatrix}.$$

On the other hand, keeping in mind that $J^T = J$ we have

$$cs(Q)^T = \begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix}^T = \begin{bmatrix} A^T & (JB)^T \\ (BJ)^T & (JAJ)^T \end{bmatrix} = \begin{bmatrix} A^T & B^T J \\ JB^T & JA^T J \end{bmatrix}.$$

Hence, $cs(Q^T) = cs(Q)^T$. □

Proposition 2.4. *The map cs is a bijection.*

Proof. Injectivity. Let $Z = A + jB$ and $W = C + jD$ and $W \neq Z$. From this, it follows that $A \neq C$ or $B \neq D$. Assume that $A \neq C$. Then

$$cs(Z) = \begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix} \qquad\qquad cs(W) = \begin{bmatrix} C & DJ \\ JD & JCJ \end{bmatrix}.$$

Since $A \neq C$ we have $cs(Z) \neq cs(W)$. Let now $B \neq D$ and assume that $cs(Z) = cs(W)$. Then from $cs(Z) = cs(W)$ it follows $JB = JD$. Now multiplying $JB = JD$ with J from the left implies $B = D$, which is a contradiction. We conclude that $cs(\cdot)$ is injective.

Surjectivity. Let $A \in CM_{2n}(\mathbb{R})$. By proposition 2.1 we can find matrices B and C such that (2.3) holds. But then $cs(B + jC) = A$ and since A was arbitrary, we conclude that cs is surjective. Now, injectivity and surjectivity of cs imply by definition that cs is a bijection. □

3. Correspondence of QR decompositions

Definition 3.1. Let $A \in M_n(\mathfrak{D})$. A pair (Q, R) with $Q, R \in M_n(\mathfrak{D})$ is a *QR decomposition of A over* $\mathfrak{D}$ if the following holds:

1. Q is orthogonal, i.e. $Q^T \cdot Q = Q \cdot Q^T = I$,
2. R is upper-triangular,
3. $A = Q \cdot R$.

The notion of a $m \times n$ *double-cone matrix* was introduced in [1]. Here we state the definition in block-form for the case of $H \in CM_{2n}(\mathbb{R})$.

Definition 3.2. Let $H \in CM_{2n}(\mathbb{R})$. Then H is a *double-cone matrix* iff there exist $A, B \in M_n(\mathbb{R})$ both upper-triangular such that

$$H = \begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix}.$$

Definition 3.3. Let $Z \in CM_n(\mathbb{R})$. A pair (W, Y), with $W, Y \in CM_n(\mathbb{R})$ is a *centrosymmetric QR decomposition* of Z if the following holds:

1. W is orthogonal matrix,

2. Y is double-cone matrix,

3. $Z = WY$.

The algorithm to obtain an approximation of a centrosymmetric QR decomposition of a given centrosymmetric matrix A was given in [1].

The following theorem provides one interpretation of the centrosymmetric QR decomposition, in the case of square centrosymmetric matrices of even order by establishing the equivalence of their centrosymmetric QR decomposition with the QR decomposition of the corresponding split-complex matrix.

Theorem 3.4. *(QR decomposition correspondence) Let $A \in M_n(\mathfrak{D})$. Then $(Q, R) \in M_n(\mathfrak{D}) \times M_n(\mathfrak{D})$ is a QR decomposition of A if and only if*

$$(cs(Q), cs(R)) \in CM_{2n}(\mathbb{R}) \times CM_{2n}(\mathbb{R})$$

is a centrosymmetric QR decomposition of $cs(A) \in CM_{2n}(\mathbb{R})$.

Proof. Let $(Q, R) \in M_n(\mathfrak{D}) \times M_n(\mathfrak{D})$ be a QR decomposition of A. Let $W = cs(Q)$ and $Y = cs(R)$. We have

$$W = \begin{bmatrix} Q_1 & Q_2J \\ JQ_2 & JQ_1J \end{bmatrix} \qquad\qquad Y = \begin{bmatrix} R_1 & R_2J \\ JR_2 & JR_1J \end{bmatrix}$$

Since $Q^T \cdot Q = I$ it follows that $cs(Q^T \cdot Q) = cs(Q^T)cs(Q) = cs(Q)^T cs(Q) = cs(I)$. From this we have $cs(Q)^T cs(Q) = I$ i.e. $W^T W = I$ hence W is orthogonal. Since R is upper-triangular and $R = R_1 + jR_2$, then by definition we have that both R_1 and R_2 are upper-triangular. Further, $cs(R)$ is centrosymmetric by definition. From this it follows that Y is centrosymmetric double-cone. Finally, we have $cs(A) = cs(Q \cdot R) = cs(Q)cs(R)$. Hence, $(cs(Q), cs(R))$ is a centrosymmetric QR decomposition of $cs(A)$.

Conversely, let $(W, Y) = (cs(Q), cs(R))$. If (W, Y) is a centrosymmetric QR decomposition of $cs(A)$ then $cs(A) = WY$ where W is centrosymmetric and orthogonal and Y is a double-cone matrix. From the fact that W is centrosymmetric we have (by Proposition 2.1) that

$$W = \begin{bmatrix} W_1 & W_2J \\ JW_2 & JW_1J \end{bmatrix}$$

Now the property of W being orthogonal i.e. the condition $W^TW = I$ implies

$$\begin{bmatrix} W_1^TW_1 + W_2^TW_2 & (W_1^TW_2 + W_2^TW_1)J \\ J(W_2^TW_1 + W_1^TW_2) & J(W_2^TW_2 + W_1^TW_1)J \end{bmatrix} = \begin{bmatrix} I & \\ & I \end{bmatrix}. \tag{3.1}$$

On the other hand, we have

$$Q = W_1 + jW_2 \qquad\qquad R = Y_1 + jY_2 \tag{3.2}$$

First we prove that Q is orthogonal. From (3.1) we obtain

$$\begin{aligned} Q^T \cdot Q &= (W_1 + jW_2)^T \cdot (W_1 + jW_2) \\ &= (W_1^T + jW_2^T) \cdot (W_1 + jW_2) \\ &= W_1^TW_1 + W_2^TW_2 + j(W_1^TW_2 + W_2^TW_1) \\ &= I + jO = I. \end{aligned}$$

The matrix Y is centrosymmetric and double-cone which implies

$$Y = \begin{bmatrix} Y_1 & Y_2J \\ JY_2 & JY_1J \end{bmatrix}$$

where both Y_1 and Y_2 are upper-triangular. This now implies that $cs^{-1}(Y) = Y_1 + jY_2$ is upper-triangular.

Finally, let us prove that $QR = A$. We have

$$Q \cdot R = cs^{-1}(W)cs^{-1}(Y) = cs^{-1}(WY) = cs^{-1}(cs(A)) = A.$$

We conclude that (Q, R) is a QR decomposition of A. □

Example 3.5. Let

$$A = \begin{bmatrix} 1+2j & 2+3j \\ 3+4j & 4+5j \end{bmatrix}.$$

Note that $A = W + jZ$ where

$$W = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}. \tag{3.3}$$

We have

$$cs(A) = cs(W + jZ) = \begin{bmatrix} W & ZJ \\ JZ & JWJ \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 4 & 5 & 4 \\ 4 & 5 & 4 & 3 \\ 2 & 3 & 2 & 1 \end{bmatrix}$$

By applying the CentrosymmetricQR algorithm from [1] to $cs(A)$ we obtain the approximations:

$$Q \approx \begin{bmatrix} -0.156594 & -0.106019 & -0.813126 & 0.550513 \\ 0.106019 & -0.156594 & 0.550513 & 0.813126 \\ 0.813126 & 0.550513 & -0.156594 & 0.106019 \\ 0.550513 & -0.813126 & -0.106019 & -0.156594 \end{bmatrix}$$

$$R \approx \begin{bmatrix} 4.51499 & 5.82806 & 4.41384 & 3.10078 \\ 0 & -0.525226 & -0.525226 & 0 \\ 0 & -0.525226 & -0.525226 & 0 \\ 3.10078 & 4.41384 & 5.82806 & 4.51499 \end{bmatrix}$$

Applying cs^{-1} to Q and R yields:

$$cs^{-1}(Q) \approx \begin{bmatrix} -0.156594 & -0.106019 \\ 0.106019 & -0.156594 \end{bmatrix} + j \begin{bmatrix} 0.550513 & -0.813126 \\ 0.813126 & 0.550513 \end{bmatrix}$$

$$cs^{-1}(R) \approx \begin{bmatrix} 4.51499 & 5.82806 \\ 0 & -0.525226 \end{bmatrix} + j \begin{bmatrix} 3.10078 & 4.41384 \\ 0 & -0.525226 \end{bmatrix}$$

with $A \approx cs^{-1}(Q) \cdot cs^{-1}(R)$. Now, from Theorem 3.4 we conclude that $(cs^{-1}(Q), cs^{-1}(R))$ is an approximation of a QR decomposition of A.

4. Conclusion

We introduced the standard centrosymmetric representation cs for split-complex matrices. Using this representation we proved that a QR decomposition of a square split-complex matrix A can be obtained by calculating the centrosymmetric QR decomposition introduced in [1] of its centrosymmetric matrix representation $cs(A)$.

5. Acknowledgements

The author would like to thank professor Zvonko Iljazović for reading and proposing improvements to the early draft of this paper.

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